*International Journal of Theoretical Physics, Vol. 31, No. 9, 1992*

# Brouwer–Zadeh (Fuzzy-Intuitionistic) Posets for Unsharp Quantum Mechanics

**Gianpiero Cattaneo**[1]

*Received February 10, 1992*

The partial ordered structure which plays for unsharp quantum mechanics the same role of orthomodular lattices for ordinary quantum mechanics is introduced. Differently from the unsharp case, in which one can identify quantum propositions (i.e., Hilbert space subspaces) with yes-no devices (i.e., orthogonal projections) they are tested by, in the unsharp case this identification is broken down: every quantum generalized proposition (i.e., pair of mutually orthogonal subspaces) is tested by many different yes-no devices (i.e., Hilbert space effects). The set of all quantum effects has a structure of Brouwer–Zadeh poset, canonically embeddable in a (minimal) Brouwer–Zadeh lattice, whereas the set of all quantum generalized propositions has a structure of Brouwer–Zadeh complete lattice.

A Brouwer–Zadeh poset is defined as a partially ordered structure equipped with two nonusual orthocomplementations: a regular degenerate (Zadeh or fuzzy-like) one and a weak (Brouwer or intuitionistic-like) one linked by an interconnection rule. Using these two orthocomplementations it is possible to introduce the two modal-like operators of necessity and possibility.

## 1. BROUWER–ZADEH OR FUZZY-INTUITIONISTIC POSETS

We introduce the definition of BZ-poset and the main properties of this structure [we refer to Cattaneo and Marino (1988) and Cattaneo and Nisticò (1989) for a more complete discussion on this subject].

*Definition 1.1.* A Brouwer–Zadeh (BZ)-poset (resp., lattice) is a structure

$$\langle \Sigma, 0, \leq, ', {}^{\sim} \rangle$$

[1]Dipartimento di Scienze dell'Informazione, Università di Milano, I-20133 Milan, Italy.



0020-7748/92/0900-1573$06.50/0 

defined as follows.

(a) $\Sigma$ is a nonempty set containing at least the element 0.

(b) $\leq$ is a partial order relation on $\Sigma$ with respect to which $\Sigma$ is a poset (resp., lattice) lower bounded by 0 (hence 0 is the least element of $\Sigma$).

(c) The mapping $': \Sigma \mapsto \Sigma$ is a regular degenerate (or quasi-) orthocomplementation, i.e., the following hold.

(doc-1) For every $a \in \Sigma$, $a = a''$

(doc-2) Let $a, b \in \Sigma$; then $a \leq b$ implies $b' \leq a'$

(re) Let $a, b \in \Sigma$; then $a \leq a'$ and $b' \leq b$ imply $a \leq b$

(d) The mapping $^{\sim}: \Sigma \mapsto \Sigma$ is a weak (or pseudo-) orthocomplementation, i.e., the following hold.

(woc-1) For every $a \in \Sigma$, $a \leq a^{\sim\sim}$

(woc-2) Let $a, b \in \Sigma$; then $a \leq b$ implies $b^{\sim} \leq a^{\sim}$

(woc-3) For every $a \in \Sigma$, $a \wedge a^{\sim} = 0$

(e) The two orthocomplementations are linked by the following interconnection rule.

(in) For every $a \in \Sigma$, $a^{\sim\prime} = a^{\sim\sim}$

A BZ-poset (resp., lattice) is said to have the half element iff the further following condition is satisfied.

(ha) An element $(1/2) \in \Sigma$ exists such that $(1/2) = (1/2)'$

Furthermore, we put $1 := 0' = 0^{\sim}$, so that for every $a \in \Sigma$, $a \leq 1$ (hence 1 is the greatest element of $\Sigma$). Trivially, $\forall a \in \Sigma$, $a^{\sim} \leq a'$.

The mapping $'$ is also called the Zadeh or fuzzy orthocomplementation, while the mapping $^{\sim}$ is the Brouwer or intuitionistic orthocomplementation. Hence a BZ-poset (resp., lattice) is also called a fuzzy-intuitionistic poset (resp., lattice).

If $\langle \Sigma, 0, \leq, ', ^{\sim} \rangle$ is a BZ-lattice, one can consider the algebraic structure $\langle \Sigma, 0, \wedge, \vee, ', ^{\sim} \rangle$ (where $\wedge$ and $\vee$ are the usual two lattice binary operations of meet and join, respectively). This algebraic structure is called *BZ* or *fuzzy-intuitionistic algebra*. By abuse of language, we will often denote briefly by $\Sigma$ either the BZ-poset or the BZ-algebra above. Anyway, as shown by (Giuntini, 1990) it is possible to embed any BZ poset in a BZ complete lattice, preserving "inf" and "sup" operations, with a standard Mac Neille completion.

## 1.1. Some Remarks on Fuzzy Orthocomplementation

The 0-*kernel* and the 1-*kernel* of the fuzzy orthocomplementation $'$ are respectively the sets

$$N'_0 := \{a \in \Sigma : a \leq a'\} \qquad \text{and} \qquad N'_1 := \{b \in \Sigma : b' \leq b\}$$

The elements of $N'_0$ are called *contingent* or of *type* II, and the elements of $N'_1$ *possible* or of *type* I. Of course, $0 \in N'_0$ and $1 \in N'_1$. Moreover, if $a \wedge a'$ exists, then $a \wedge a' \in N'_0$ and if $b \vee b'$ exists, then $b \vee b' \in N'_1$. Furthermore, $a \in N'_0$ iff $a' \in N'_1$. Finally, the *central kernel* of $'$ is defined as the set

$$N'_c := N'_0 \cap N'_1 = \{a \in \Sigma : a = a'\}$$

whose elements are said to be the *central* elements.

In a degenerate orthocomplemented poset the following conditions are equivalent:

(doc-3a) $N'_0 = \{0\}$
(doc-3b) $N'_1 = \{1\}$
(doc-3c) For every $a \in \Sigma$, $a \wedge a' = 0$
(doc-3d) For every $a \in \Sigma$, $a \vee a' = 1$

If one, and therefore all, of the aforesaid conditions holds, the fuzzy orthocomplementation $'$ is said to be *standard* and the bounded poset $\Sigma$ is called *orthocomplemented* tout court. Condition (doc-3c) expresses the *contradiction law*, whereas condition (doc-3d) is the *excluded-middle* law of the orthocomplementation.

As customary, from any orthocomplementation (also if nonusual) it is possible to define an *orthogonality* relation $\perp$ on $\Sigma$ as follows. Let $a, b \in \Sigma$;

(og) $a \perp b$ iff $a \leq b'$

Relation $\perp$ satisfies the following conditions. Let $a, b, c \in \Sigma$; then the following hold.

(og-1) $a \perp b$ iff $b \perp a$
(og-2) $0 \perp a$
(og-3) $a \leq b$ and $b \perp c$ imply $a \perp c$
(og-4) $a \perp \{c : c \perp b\}$ implies $a \leq b$
(og-5) $a' \perp a$ and $a' = \vee\{b : b \perp a\}$

The fuzzy orthocomplementation satisfies both the generalized de Morgan laws:

(dM-1d) Let $a, b \in \Sigma$; if $a \vee b$ exists in $\Sigma$; then $a' \wedge b'$ exists in $\Sigma$, too, and $a' \wedge b' = (a \vee b)'$

(dM-2d) Let $a, b \in \Sigma$; if $a \wedge b$ exists in $\Sigma$, then $a' \vee b'$ exists in $\Sigma$, too, and $a' \vee b' = (a \wedge b)'$

More precisely, it is possible to prove that if a mapping on a poset satisfies condition (doc-1), then statements (doc-2), (dM-1d), and (dM-2d) are equivalent. On the contrary, the intuitionistic orthocomplementation generally satisfies only the first generalized de Morgan law.

(dM-1w) Let $a, b \in \Sigma$; if $a \vee b$ exists in $\Sigma$, then $a^{\sim} \wedge b^{\sim}$ exists in $\Sigma$, too, and $a^{\sim} \wedge b^{\sim} = (a \vee b)^{\sim}$

Furthermore, if $\Sigma$ is a BZ-lattice, condition (re) is equivalent to the following *Kleene* condition.

(KL) For every $a, b \in \Sigma$, $a \wedge a' \leq b \vee b'$

If $\Sigma$ is BZ-poset with half element, then $(1/2) \in N'_c$; moreover, conditions (ha) and (re) can be equivalent replaced by the following statements:

(ha-1) $N'_c = \{(1/2)\}$ is a singleton
(re-1) $a \leq (1/2)$ for every $a \in N'_0$ and $(1/2) \leq b$ for every $b \in N'_1$

### 1.2. Exact (Sharp) and Fuzzy (Unsharp) Elements of a BZ-Poset

Let $\langle \Sigma, 0, \leq, ', ^{\sim} \rangle$ be a BZ-poset. We introduce the set

$$\Sigma_e := \{f \in \Sigma : f = f^{\sim\sim}\}$$

of all the elements of $\Sigma$ which are closed with respect to the Brouwerian orthocomplementation. The elements of $\Sigma_e$ are called *exact* (or *sharp*), and the elements of $\Sigma \backslash \Sigma_e$ *fuzzy* (or *unsharp*).

*Theorem 1.1.* In any BZ-poset $\Sigma$ the elements $0, 1 \in \Sigma_e$. Moreover, the following statements hold.

(i) For every $f \in \Sigma_e, f' = f^{\sim} \in \Sigma_e$.

(ii) Let us still denote by $\leq$ the restriction to $\Sigma_e$ of the partial order relation defined on $\Sigma$; then, the mapping

$$f \in \Sigma_e \to f' = f^{\sim} \in \Sigma_e$$

is a standard orthocomplementation in $\langle \Sigma_e, 0, \leq \rangle$, which we still denote by $'$, so that $\langle \Sigma_e, 0, \leq, ' \rangle$ is a (standard) orthocomplemented bounded poset.

(iii) If $\Sigma$ is a lattice, then $\Sigma_e$ is a sublattice of $\Sigma$, whose lattice l.u.b. and g.l.b., denoted by $\vee_e$ and $\wedge_e$, respectively, are just the l.u.b. and the g.l.b. in $\Sigma$, i.e., for any $f, g \in \Sigma_e, f \vee_e g = f \vee g$ and $f \wedge_e g = f \wedge g$.

(iv) For every $a \in \Sigma$, $a^{\sim\prime}$ and $a'^{\sim}$ are exact elements of $\Sigma$.

## 2. NECESSITY AND POSSIBILITY

Resting on (iv) in Theorem 1.1, two further unary operators can be introduced in every BZ-poset:

$$\nu: \quad a\in\Sigma \mapsto \nu(a) := a'^{\sim} \in \Sigma_e \qquad \text{(necessity)}$$

$$\mu: \quad a\in\Sigma \mapsto \mu(a) := a^{\sim\prime} \in \Sigma_e \qquad \text{(possibility)}$$

The following are some elementary properties of the operators $\nu, \mu$. For every $a\in\Sigma$,

$$\nu(a') = \mu(a)', \qquad \mu(a') = \nu(a)', \qquad \nu(a^{\sim}) = \mu(a)^{\sim}$$

Moreover, $\nu$ and $\mu$ act on the exact elements of $\Sigma_e$ as identity operators, i.e., for every $f\in\Sigma_e$, $\nu(f) = \mu(f) = f$ [hence $\Sigma_e = \nu(\Sigma) = \mu(\Sigma)$].

*Definition 2.1.* A BZ-poset $\Sigma$ is said to be a quantum (resp., classical) BZ-poset iff the following conditions are satisfied:

(Q1) The orthocomplemented poset of all exact elements $\Sigma_e$ is an orthomodular (resp., Boolean) lattice.

(Q2) For any family of elements $\{a_j\} \subseteq \Sigma$, there exists an element $\pi a_j \in \Sigma$ such that

$$\nu(\pi a_j) = \wedge_j\, \nu(a_j) \qquad \text{and} \qquad \mu(\pi a_j) = \wedge_j\, \mu(a_j)$$

*Theorem 2.1.* The necessity operator $\nu: \Sigma \mapsto \Sigma_e$ satisfies the following.

| | | |
|---|---|---|
| (in-1) | $\nu(1) = 1$ | (normalized) |
| (in-2) | $\nu(a) \leq a$ | (decreasing) |
| (in-3) | $a \leq b$ implies $\nu(a) \leq \nu(b)$ | (monotone) |
| (in-4) | $\nu\nu(a) = \nu(a)$ | (idempotent) |
| (uq) | $\nu(\nu(a)') = \nu(a)'$ | (interconnection) |
| (ne-1) | $\nu(a) \wedge \nu(a)' = 0$ | (contradiction) |
| (ne-2) | $\nu(a) \vee \nu(a)' = 1$ | (excluded-middle) |

Furthermore, the following also holds.

(n-1) $a \wedge \nu(a') = 0$

Conditions (in-1)–(in-4) define $\nu$ as an *interior operator*; because of condition (uq), this interior operator is a *universal quantifier*. Conditions (ne-1) and (ne-2) express the *modal principles* of *contradiction* and *excluded middle* for the necessity operator, respectively.

Then, the following theorem can be stated.

*Theorem 2.2.* Let $\langle \Sigma, 0, \leq, ', \nu \rangle$ be any regular degenerate orthocomplemented poset endowed with a necessity operator $\nu$, i.e., a mapping on

$\Sigma$ which satisfies (in-1)–(in-3), (uq), and (n-1) in Theorem 2.1. Then, the mapping

$$^{\sim}: \quad a\in\Sigma \mapsto a^{\sim} := \nu(a')\in\Sigma$$

is a weak orthocomplementation, the structure $\langle\Sigma, 0, \leq, ', ^{\sim}\rangle$ is a BZ-poset, and the necessity operator induced by $^{\sim}$ coincides with $\nu$.

Because of Theorem 2.2, the structure $\langle\Sigma, 0, \leq, ', \nu\rangle$, with a necessity operator on $\Sigma$, is also called a BZ-poset.

In an analogous way as for necessity, we have the following result.

*Theorem 2.3.* The possibility operator $\mu : \Sigma \mapsto \Sigma_e$ satisfies the following.

| | | |
|---|---|---|
| (cl-1) | $\mu(0)=0$ | (normalized) |
| (cl-2) | $a\leq\mu(a)$ | (increasing) |
| (cl-3) | $a\leq b$ implies $\mu(a)\leq\mu(b)$ | (monotone) |
| (cl-4) | $\mu\mu(a)=\mu(a)$ | (idempotent) |
| (eq) | $\mu(\mu(a)')=\mu(a)'$ | (interconnection) |
| (po-1) | $\mu(a)\wedge\mu(a)'=0$ | (contradiction) |
| (po-2) | $\mu(a)\vee\mu(a)'=1$ | (excluded-middle) |

Furthermore, the following also holds.

(p-1) $a\wedge\mu(a)'=0$

Conditions (cl-1)–(cl-4) define $\mu$ as a *closure operator*. Because of condition (eq), this closure operator is an *existential quantifier*. Conditions (po-1) and (po-2) express the modal principles of contradiction and excluded-middle for the possibility operator.

Then, the following theorem can be stated.

*Theorem 2.4.* Let $\langle\Sigma, 0, \leq, ', \mu\rangle$ be any regular degenerate orthocomplemented poset endowed with a possibility operator $\mu$, i.e., a mapping on $\Sigma$ which satisfies (cl-1)–(cl-3), (eq), and (p-1) in Theorem 2.3. Then, the mapping

$$^{\sim}: \quad a\in\Sigma \mapsto a^{\sim} := \mu(a)'\in\Sigma$$

is a weak orthocomplementation, the structure $\langle\Sigma, 0, \leq, ', ^{\sim}\rangle$ is a BZ-poset, and the possibility operator induced by $^{\sim}$ coincides with $\mu$.

Because of Theorem 2.4, the structure $\langle\Sigma, 0, \leq, ', \mu\rangle$, with $\mu$ a possibility operator on $\Sigma$, is also called a BZ-poset.

### 2.1. The Weak Anti-intuitionistic Orthocomplementation

The mapping

$$^\flat: \quad a\in\Sigma \mapsto a^\flat := a'^{\sim\prime} \in \Sigma_e$$

is a weak anti-orthocomplementation, or anti-pseudo-orthocomplementation, i.e., it satisfies the following conditions.

(aoc-1) For every $a\in\Sigma$, $a^{\flat\flat}\leq a$
(aoc-2) Let $a, b\in\Sigma$; then $a\leq b$ implies $b^\flat \leq a^\flat$
(aoc-3) For every $a\in\Sigma$, $a\vee a^\flat = 1$

The connection between $^\flat$, $^\sim$, $'$ is established as follows:

$$\text{for every } a\in\Sigma, \qquad a^\sim \leq a' \leq a^\flat$$

Finally, for every $a\in\Sigma$, the following equalities hold:

$$a^\flat = \nu(a)' = \nu(a)^\sim = \nu(a)^\flat \qquad \text{(nonnecessity)}$$

$$a^\sim = \mu(a)' = \mu(a)^\sim = \mu(a)^\flat \qquad \text{(impossibility)}$$

## 3. BZ PROPERTIES AND NOPERTIES

We denote by $\equiv$ the equivalence relation on $\Sigma$ defined as follows. Let $a, b\in\Sigma$; then

$$a\equiv b \qquad \text{iff} \quad \nu(a)=\nu(b)$$

Whenever $a\equiv b$ we say that $a$ and $b$ are equivalent with respect to necessity. We call *property*, written $pr(a)$, the equivalence class with respect to $\equiv$ generated by $a\in\Sigma$; the collection of all such properties is denoted by $pr(\Sigma)$.

The element $\nu(a)$ belongs to $pr(a)$, it is the unique exact element of $pr(a)$, and for all other $b\in pr(a)$ we have that $\nu(a)\leq b$. For these reasons $\nu(a)$ is called the *exact representative* of $pr(a)$; all the other elements in the same property are said to be *fuzzy representatives* of it.

We can identify the property $pr(a)$ with its exact representative $\nu(a)$, according to

$$pr(a) \leftrightarrow \nu(a) \tag{3.1}$$

The set $pr(\Sigma)$ of all the properties is an orthocomplemented bounded poset, with respect to the order relation,

(or-pr) For every $a, b\in\Sigma$, $\quad pr(a)\leq pr(b) \qquad \text{iff} \qquad \nu(a)\leq \nu(b)$

and the standard orthocomplementation

(oc-pr) For every $a \in \Sigma, pr(a)' = pr(v(a)')$

In conclusion, we schematically summarize all this in the following way:

$$\begin{array}{c} \Sigma \\ \cup\!| \\ v(a) \in \Sigma_e \equiv pr(\Sigma) \ni pr(a) \end{array} \tag{3.2}$$

We denote by $\equiv_0$ the equivalence relation on $\Sigma$ defined as follows. Let $a, b \in \Sigma$; then

$$a \equiv_0 b \qquad \text{iff} \quad \mu(a) = \mu(b)$$

Whenever $a \equiv_0 b$ we say that $a$ and $b$ are equivalent with respect to possibility. We denote by $pr_0(a)$ the equivalence class with respect to $\equiv_0$ generated by $a \in \Sigma$. Furthermore, the unique exact element $\mu(a)$ which belongs to $pr_0(a)$ is called the *exact representative* of this class; all the other elements in $pr_0(a)$ are its *fuzzy representatives*.

Now, let us observe that we can make the following identifications

$$pr_0(a) \leftrightarrow \mu(a) \tag{3.3}$$

$$pr_0(a) \leftrightarrow \mu(a)' \tag{3.4}$$

Therefore, the equivalence class $pr_0(a)$ can be identified either with $\mu(a)$ or with $\mu(a)'$ (which both are exact elements), thus obtaining two semantically different interpretations of the same class.

> Whenever the equivalence class $pr_0(a)$ is represented by the unique exact element $a^{\sim} = \mu(a)'$, i.e., the *impossibility* of $a$, we refer to it as to a *noperty*.

Each of the above identifications induces an order and a standard orthocomplementation on $pr_0(\Sigma)$ in a way similar to the one we have discussed in the case of properties.

*Remark 1.* It must be noted that $f$ is an exact representative of a property iff $f'$ is an exact representative of a noperty and vice versa.

## 4. THREE-VALUED BZ-POSETS INDUCED FROM BZ-POSETS BY MODALITY

From any BZ-poset it is possible to construct a new BZ-poset, according to a procedure expounded in the present section.

*Definition 4.1.* Let $\langle \Sigma, 0, \leq, ', ^{\sim} \rangle$ be a BZ-poset. Then, we put

$$ext\colon \quad a \in \Sigma \mapsto ext(a) := (v(a), v(a')) \in \Sigma_e \times \Sigma_e$$

The mapping *ext* is called the extension mapping; it associates to any element $a \in \Sigma$ the corresponding extension $(\nu(a), \nu(a'))$. The necessity $\nu(a)$ and the impossibility $\nu(a') = \mu(a)'$ are said to be the certainly-true and certainly-false element of $ext(a)$, respectively.

Trivially, for every $f \in \Sigma_e$, $ext(f) = (f, f')$. Furthermore, we put $L_3(\Sigma) = ext(\Sigma) = \{(\nu(a), \nu(a')) : a \in \Sigma\}$ and we state the following theorem.

*Theorem 4.1.* The structure $\langle L_3(\Sigma), |0|, \subseteq, ^-, \sim \rangle$, where $|0| := (0, 1)$, is a BZ-poset with respect to the following.

(1) The partial ordering:

(or-L) $(\nu(a), \nu(a')) \subseteq (\nu(b), \nu(b'))$ iff $\nu(a) \leq \nu(b)$ and $\nu(b') \leq \nu(a')$.

(2) The (regular) degenerate orthocomplementation:

(doc-L) $-(\nu(a), \nu(a')) := (\nu(a'), \nu((a')')) = (a^{\sim}, \nu(a))$.

(3) The weak orthocomplementation:

(woc-L) $\sim(\nu(a), \nu(a')) := (\nu(a^{\sim}), \nu((a^{\sim})')) = (a^{\sim}, \mu(a))$.

This BZ-poset will be called the *three-valued BZ*, or *Łukasiewicz*, *poset induced from* $\Sigma$ *by modality.*

The following proposition collects some elementary properties of the Łukasiewicz poset defined above.

*Proposition 4.1.*

(i) The Łukasiewicz BZ-poset $L_3(\Sigma)$ is bounded by the least element $|0| = (0, 1)$ and the greatest element $|1| = (1, 0)$.

(ii) Whenever $\Sigma$ has the half element $(1/2)$, then the element

$$|(1/2)| := (\nu(1/2), \nu((1/2)')) = (0, 0)$$

is the half element of $L_3(\Sigma)$.

(iii) The kernels of the degenerate orthocomplementation $-$ are

$$N_0^- = \{(\nu(a), \nu(a')) : \nu(a) = 0\}, \qquad N_1^- = \{(\nu(b), \nu(b')) : \nu(b') = 0\}$$

(iv) The exact elements in $L_3(\Sigma)$ are generated by the exact elements in $\Sigma_e$; more precisely,

$$L_3(\Sigma_e) = \{(f, f') : f \in \Sigma_e\} = ext(\Sigma_e)$$

(v) The necessity and possibility operators of the BZ-poset $L_3(\Sigma)$, which we now denote by $\Box$ and $\Diamond$, respectively, are defined by the following equalities.

$$\Box(\nu(a), \nu(a')) = (\nu(a), \nu(a)') \qquad \text{(necessity-L)}$$

$$\Diamond(\nu(a), \nu(a')) = (\mu(a), \mu(a)') \qquad \text{(possibility-L)}$$

The elements from $L_3(\Sigma)\backslash L_3(\Sigma)_e$ are called *modal iper fuzzy* or *m-iper fuzzy*, for short; the elements from $L_3(\Sigma)_e$ are called *m-iper exact*.

Notice that from Proposition 4.1(iv) the restriction of the extension mapping to $\Sigma_e$ is one-to-one and onto $L_3(\Sigma)_e$, allowing the identification between m-iper exact elements from $L_3(\Sigma)_e$ and exact elements from $\Sigma_e$ according to

$$(f, f') \leftrightarrow f \tag{4.1}$$

Schematically, we represent all this in the following diagram:

$$\begin{array}{c} L_3(\Sigma) \\ \cup\!\shortmid \\ (f, f') \in L_3(\Sigma)_e \equiv \Sigma_e \ni f \end{array} \tag{4.2}$$

The property from $pr(L_3(\Sigma))$ generated by $(\nu(a), \nu(a'))$ is

$$pr(\nu(a), \nu(a')) = \{(\nu(b), \nu(b')) : \nu(a) = \nu(b)\}$$

From Proposition 4.1(v) and (4.1) it follows that the identification (3.1) between properties and exact elements representing properties can be extended in the following way:

$$pr(\nu(a), \nu(a')) \overset{(3.1)}{\longleftrightarrow} (\nu(a), \nu(a)') \overset{(4.1)}{\longleftrightarrow} \nu(a) \tag{4.3}$$

Analogously, we get

$$pr_0(\nu(a), \nu(a')) \overset{(3.4)}{\longleftrightarrow} (\mu(a)', \mu(a)) \overset{(4.1)}{\longleftrightarrow} \mu(a)' \tag{4.4}$$

With the above identifications, the certainly-true element $\nu(a)$ and the certainly-false element $\nu(a') = \mu(a)'$ of $ext(a) = (\nu(a), \nu(a'))$ are the exact representatives of the property and of the noperty generated by $ext(a)$, respectively.

If in (4.3) and (4.4) we take once more into account identifications (3.1) and (3.4), we can further identify the corresponding property sets according to

$$pr(\nu(a), \nu(a')) \overset{(4.3)}{\longleftrightarrow} \nu(a) \overset{(3.1)}{\longleftrightarrow} pr(a) \tag{4.5}$$

$$pr_0(\nu(a), \nu(a')) \overset{(4.4)}{\longleftrightarrow} \mu(a)' \overset{(3.4)}{\longleftrightarrow} pr_0(a) \tag{4.6}$$

In conclusion, we sum up the identifications up to now in the scheme

$$\begin{array}{ccc} \Sigma & \xrightarrow{\text{ext}} & L_3(\Sigma) \\ \cup\!| & & \cup\!| \\ pr(\Sigma)\equiv\Sigma_e & \equiv & L_3(\Sigma)_e \equiv pr(L_3(\Sigma)) \end{array} \tag{4.7}$$

which can be made explicit in the following way:

$$\begin{array}{ccccccc} & & a & \xrightarrow{\text{ext}} & (\nu(a),\, \nu(a')) & & \\ & & {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \Box} & & \\ pr(a) & \longleftrightarrow & \nu(a) & \longleftrightarrow & (\nu(a),\, \nu(a)') & \longleftrightarrow & pr(\nu(a),\, \nu(a')) \end{array} \tag{4.8}$$

where [see conditions (in-4) and (uq) of Theorem 2.1] the extension mapping

$$ext(a) = (\nu(a),\, \nu(a)') = (\nu(\nu(a)),\, \nu(\nu(a)')) \in L_3(\Sigma)$$

in general is not one-to-one. The $L_3(\Sigma)$ three-valued BZ poset induced from a BZ-poset $\Sigma$ by modality, in general, is not a lattice. The following theorem gives a deepening of the structure of a quantum BZ-poset.

*Theorem 4.2.* Let $\Sigma$ be a quantum BZ-poset; then $L_3(\Sigma)$ is a BZ complete lattice, where, in particular, the following hold.

(l-1) $(\nu(a),\, \nu(a')) \cap (\nu(b),\, \nu(b')) = (\nu(a) \wedge \nu(b),\, \nu(a') \vee \nu(b'))$
(l-2) $(\nu(a),\, \nu(a')) \cup (\nu(b),\, \nu(b')) = (\nu(a) \vee \nu(b),\, \nu(a') \wedge \nu(b'))$

## 5. GENERALIZED PROPERTIES

Let $\Sigma$ be a BZ-poset. The following is an equivalence relation on $\Sigma$:

$$\begin{aligned} a \cong b \quad & \text{iff} \quad ext(a) = ext(b) \\ & \text{iff} \quad \nu(a) = \nu(b) \quad \text{and} \quad \mu(a) = \mu(b) \end{aligned}$$

This equivalence relation is called *congruence by modality*. We will denote by $|a|$ the congruence class generated by element $a \in \Sigma$. Any such equivalence class is called a *generalized property* and we put $pr_g(\Sigma) := (\Sigma/\cong)$.

Since for any $a \in \Sigma$

$$pr(a) = \{c \in \Sigma\colon \nu(a) = \nu(c)\}, \qquad pr_0(a) = \{d \in \Sigma\colon \mu(a) = \mu(d)\}$$

we have

$$|a| = \{b\colon \nu(a) = \nu(b),\, \mu(a) = \mu(b)\} = pr(a) \cap pr_0(a)$$

The congruence relation $\cong$ is compatible with operations of orthocomplementation $'$ and $^\sim$; that is,

$$a \cong b \quad \text{implies} \quad a' \cong b' \quad \text{and} \quad a^\sim \cong b^\sim$$

and from this result we get the following theorem.

*Theorem 5.1.* The set of all the generalized properties

$$\langle pr_g(\Sigma), |0|, \subseteq, -, \sim \rangle$$

is a BZ-poset once equipped with the following.

(1) The partial ordering

(or-P) $|a| \subseteq |b|$ iff $\nu(a) \leq \nu(b)$ and $\mu(a) \leq \mu(b)$.

(2) The degenerate orthocomplementation:

(doc-P) $-: pr_g(\Sigma) \mapsto pr_g(\Sigma), |a| \to -|a| := |a'|$.

(3) The weak orthocomplementation:

(woc-P) $\sim: pr_g(\Sigma) \mapsto pr_g(\Sigma), |a| \to -|a| := |a^\sim|$.

The mapping $Ext: pr_g(\Sigma) \mapsto L_3(\Sigma)$, associating to any generalized property $|a|$ its three-valued "extension"

$$Ext(|a|) := (\nu(a), \nu(a')) = (\nu(a), \mu(a)')$$

is an order isomorphism which preserves the two orthocomplementation mappings and which allows the identification

$$\begin{aligned} pr_g(\Sigma) &\equiv L_3(\Sigma) \\ |a| &\longleftrightarrow (\nu(a), \nu(a')) \end{aligned} \tag{5.1}$$

*Proposition 5.1.* Denote by $pr_e(\Sigma)$ the set of all exact elements of the BZ-poset $pr_g(\Sigma)$; the following statements hold.

(i) The generalized property $|a|$ is exact in $pr_g(\Sigma)$ iff the element $a$ is exact in $\Sigma$, i.e.,

$$|a| \in pr_e(\Sigma) \quad \text{iff} \quad a \in \Sigma_e$$

(ii) $pr_e(\Sigma)$ is the collection of all the congruence classes of the type $|\nu(a)|$, for $a$ running in $\Sigma$, i.e.,

$$pr_e(\Sigma) = \{|\nu(a)| : a \in \Sigma\}$$

(iii) For every $a \in \Sigma$, $|\nu(a)|$ is the singleton

$$|\nu(a)| = \{\nu(a)\}$$

The elements from $pr_e(\Sigma)$ are called *exact properties*, whereas the ones from $pr_g(\Sigma)\backslash pr_e(\Sigma)$ are called *fuzzy properties*, in the sequel.

*Proof.* $\sim(\sim|a|)=|a|$ iff $|a^{\sim\sim}|=|a|$ iff $\nu(a^{\sim\sim})=\nu(a)$ and $\mu(a^{\sim\sim})=\mu(a)$, but, since $\nu(a^{\sim\sim})=\mu(a^{\sim\sim})$, this is true if $\nu(a)=\mu(a)$ and from (in-2) and (cl-2) we get $a=\nu(a)=\mu(a)$. This proves (i) and (ii). Now, $b\in|\nu(a)|$ iff $\nu(\nu(a))=\nu(b)$ and $\mu(\nu(a))=\mu(b)$; since $\nu(\nu(a))=\nu(a)$ and $\mu(\nu(a))=\nu(a)$, we conclude that $\nu(b)=\mu(b)=\nu(a)$, and from (in-2) and (cl-2) we conclude that $b=\nu(a)$, which proves (iii).

Note that as a particular case of (5.1) we have the identification

$$|\nu(a)| \overset{(5.1)}{\longleftrightarrow} (\nu(\nu(a)),\, \nu(\nu(a)')) \overset{(\mathrm{uq})}{=} (\nu(a),\, \nu(a)')$$

According to the above proposition, this identification is schematically represented by the diagram

$$\begin{aligned} &pr_e(\Sigma)\equiv L_3(\Sigma)_e \\ &|\nu(a)| \leftrightarrow (\nu(\nu(a)),\, \nu(\nu(a'))) \end{aligned} \tag{5.2}$$

We can now put together these results with (4.7) and (4.8) to get the following diagram:

$$\begin{array}{ccccc} & \Sigma & \overset{\mathrm{ext}}{\longrightarrow} & L_3(\Sigma) \equiv pr_g(\Sigma) & \\ & \cup\!| & & \cup\!| \qquad\quad \cup\!| & \\ pr(\Sigma)\equiv & \Sigma_e & \equiv & L_3(\Sigma)_e \equiv pr_e(\Sigma) & \end{array} \tag{5.3}$$

with extended behavior

$$\begin{array}{ccccccc} & a & \overset{\mathrm{ext}}{\longrightarrow} & (\nu(a),\, \nu(a')) & \longleftrightarrow & |a| \\ & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \Box} & & \downarrow \\ pr(a) \longleftrightarrow & \nu(a) & \longleftrightarrow & (\nu(a),\, \nu(a)') & \longleftrightarrow & |\nu(a)| \end{array} \tag{5.4}$$

where

$$|a|=\{b\in\Sigma\colon \nu(a)=\nu(b) \quad \text{and} \quad \mu(a)=\mu(b)\}$$

$$|\nu(a)|=\{\nu(a)\} \quad \text{and} \quad pr(a)=\{b\in\Sigma\colon \nu(a)=\nu(b)\} \quad \blacksquare$$

## 6. THE HILBERTIAN BZ-POSET

Let $\Sigma(\mathscr{H})$ denote the set of all bounded self-adjoint operators $F$ defined on a complex Hilbert space $\mathscr{H}$, whose inner product is $\langle\cdot|\cdot\rangle$, and satisfying the condition

$$\forall\varphi\in\mathscr{H}, \qquad 0\leq\langle F\varphi|\varphi\rangle\leq\|\varphi\|^2 \tag{6.1}$$

The set $\mathscr{E}(\mathscr{H})$ of all orthogonal projections is enclosed in this class. Let $\mathscr{M}(\mathscr{H})$ denote the set of all subspaces (i.e., closed linear manifolds) of $\mathscr{H}$;

then the mapping $\mathcal{M}(\mathcal{H}) \to \mathcal{E}(\mathcal{H})$, $M \to E_M$, associating to the subspace $M$ of $\mathcal{H}$ the orthogonal projection $E_M$ which projects onto $M$, is one-to-one and onto, allowing the following identification:

$$M \leftrightarrow E_M \tag{6.2}$$

*Theorem 6.1.* The set $\Sigma(\mathcal{H})$ is a BZ-poset $\langle \Sigma(\mathcal{H}), \mathbb{O}, \leq, ', ^{\sim} \rangle$ (which in general is not a lattice) with respect to the following:

(1) The partial ordering

(or-H) $F_1 \leq F_2$ iff $\forall \varphi \in \mathcal{H}, \langle F_1\varphi|\varphi\rangle \leq \langle F_2\varphi|\varphi\rangle$.

(2) The (regular) degenerate orthocomplementation:

(doc-H) $F' := \mathbb{1} - F$.

(3) The weak orthocomplementation:

(woc-H) $F^{\sim} := E_{\mathrm{Ran}(F)^{\perp}} = E_{\ker(F)}$.

We collect now some interesting properties of this BZ-poset.

*Proposition 6.1.* The BZ-poset $\Sigma(\mathcal{H})$ satisfies the following statements.

(i) It is bounded by the least operator "$\mathbb{O}(\varphi) = \underline{0}$, for all $\varphi \in \mathcal{H}$" ($\underline{0}$ being the zero vector of $\mathcal{H}$) and the greatest operator "$\mathbb{1}(\varphi) = \varphi$, for all $\varphi \in \mathcal{H}$," i.e.,

$$\mathbb{O} \leq F \leq \mathbb{1}$$

(ii) The operator "$(\frac{1}{2}\mathbb{1})(\varphi) := \frac{1}{2}\varphi$, for all $\varphi \in \mathcal{H}$" is such that $(\frac{1}{2}\mathbb{1})' = (\frac{1}{2}\mathbb{1})$. Then this operator is the half element of $\Sigma(\mathcal{H})$.

(iii) The kernels of the degenerate orthocomplementation are

$$N'_0 = \{F \in \Sigma(\mathcal{H}) : F \leq (\tfrac{1}{2}\mathbb{1})\}, \qquad N'_1 = \{F \in \Sigma(\mathcal{H}) : (\tfrac{1}{2}\mathbb{1}) \leq F\}$$

(iv) The family $\Sigma_e(\mathcal{H})$ of all exact elements in $\Sigma(\mathcal{H})$ is the set $\mathcal{E}(\mathcal{H})$ of all orthogonal projections on $\mathcal{H}$.

(v) The necessity and possibility operators are given by

$$\nu(F) = F'^{\sim} = (\mathbb{1} - F)^{\sim} = E_{\mathrm{Ker}(\mathbb{1}-F)} \qquad \text{(necessity)}$$

$$\mu(F) = F^{\sim\prime} = E_{\mathrm{Ker}(F)^{\perp}} \qquad \text{(possibility)}$$

from which we have

$$\mu(F)' = F^{\sim} = E_{\mathrm{Ker}(F)} \qquad \text{(impossibility)}$$

By making use of the usual nomenclature of BZ-poset theory, the elements of $\Sigma(\mathcal{H})$ are said to be *generalized orthogonal projections*. The generalized orthogonal projections which are not orthogonal projections are called *fuzzy projections*, whereas the orthogonal projections are also called *exact projections* [see (iv)].

To any generalized orthogonal projection $F \in \Sigma(\mathscr{H})$ it is possible to associate two subspaces of $\mathscr{H}$:

The *certainly-yes domain*: $M_1(F) = \{\psi \in \mathscr{H} : \langle F\psi|\psi\rangle = \|\psi\|^2\}$.
The *certainly-no domain*: $M_0(F) = \{\phi \in \mathscr{H} : \langle F\phi|\phi\rangle = 0\}$.

It is a trivial result of the Hilbert space operator theory that

$$M_1(F) = \mathrm{Ker}(\mathbb{1} - F) \qquad \text{and} \qquad M_0(F) = \mathrm{Ker}(F) \tag{6.3}$$

Generalized orthogonal projections are Hilbertian representatives of (Ludwig's) *effects* (Ludwig, 1983; Kraus, 1983) produced in *yes–no measurement devices* testing *questions* of Piron's approach to quantum physics (Jauch and Piron, 1969; Piron, 1976; also see Cattaneo and Nisticò, 1985; Cattaneo *et al.*, 1988, 1989). More precisely:

(RI-1) Fuzzy projections from $\Sigma(\mathscr{H}) \backslash \mathscr{E}(\mathscr{H})$ are interpreted as *fuzzy* (or *unsharp*) *effects* (produced in yes–no measurement devices) and orthogonal projections from $\mathscr{E}(\mathscr{H})$ as *exact* (or *sharp*) *effects* (also *events*).

(RI-2) Vectors of $\mathscr{H} \backslash \{\underline{0}\}$ are interpreted as *preparation apparatuses* of ensembles of noninteracting individual samples of the physical system, similarly prepared under well-defined and repeatable conditions.

(RI-3) The quantity, defined for any $F \in \Sigma(\mathscr{H})$ and any $\varphi \in \mathscr{H} \backslash \{\underline{0}\}$

$$P(\varphi, F) := \frac{\langle F\varphi|\varphi\rangle}{\|\varphi\|^2}$$

is such that $0 \leq P(\varphi, F) \leq 1$; hence it is interpreted as the *probability of occurrence of* the answer "yes" for the effect $F$ when the system is prepared according to $\varphi$.

(RI-4) Nonzero vectors of the certainly-yes (resp., no) domain $M_1(F)$ [resp., $M_0(F)$] represent preparation apparatuses with respect to which the answer yes (resp., no) to the effect $F$ (resp., $F'$) is certain, i.e., the probability of occurrence of $F$ (resp., $F'$) is 1. When $F$ (resp., $F'$) is certain, we also say that $F$ is *true* (resp., *false*).

Owing to Proposition 6.1(v) and (6.3), the necessity $\nu(F)$ of any generalized orthogonal projection $F$ is the orthogonal projection $E_{M_1(F)}$ which projects onto the certainly-yes domain of $F$; therefore, two generalized orthogonal projections $F_1$ and $F_2$ are equivalent with respect to necessity iff they have the same certainly-yes domain:

$$F_1 \equiv F_2 \quad \text{iff} \quad E_{M_1(F_1)} = E_{M_1(F_2)}$$
$$\quad \text{iff} \quad M_1(F_1) = M_1(F_2)$$

Any necessity equivalence class, in this quantum context denoted by

$$[F]_{\text{JP}} = \{F_j \colon F \equiv F_j\} = \{F_j \colon M_1(F) = M_1(F_j)\}$$

defines a (quantum) *property* of the physical system; this property is characterized by the certainly-yes domain $M_1(F)$ common to any generalized orthogonal projections (i.e., $F_j$) belonging to it [i.e., $M_1(F_j) = M_1(F)$]. The exact effect described by the orthogonal projection $E_{M_1(F)}$ belongs to the property $[F]_{\text{JP}}$, it is the unique orthogonal projection belonging to $[F]_{\text{JP}}$, and $E_{M_1(F)} \leq F_j$ for all other $F_j \in [F]_{\text{JP}}$. In this Hilbertian BZ-poset, the (3.1) has now the form

$$[F]_{\text{JP}} \leftrightarrow E_{M_1(F)} \tag{6.4}$$

Owing to this identification, in axiomatic quantum mechanics $E_M$ is customarily assumed to represent the exact effect produced in yes–no measurement devices which *sharply*, i.e., without noise and imprecision, test property $[F]_{\text{JP}}$. For this reason people generally identify any quantum property with the exact effect which tests it.

Any generalized orthogonal projection $F_j \in [F]_{\text{JP}}$ represents an effect produced in yes–no devices which *fuzzily* test the same quantum property $[F]_{\text{JP}}$, and so we have that a fixed quantum property is tested by several effects, breaking in this way the above identification.

Two generalized orthogonal projections $G_1$ and $G_2$ are equivalent with respect to impossibility if they have the same certainly-no domain:

$$G_1 \equiv_0 G_2 \quad \text{iff} \quad E_{M_0(G_1)} = E_{M_0(G_2)}$$
$$\quad \text{iff} \quad M_0(G_1) = M_0(G_2)$$

Any (quantum) *noperty*, denoted by $[G]_{\text{CN}}$, is characterized by a well-defined certainly-no domain $M_0(G)$ which is just the certainly-no domain of any effect belonging to it. This noperty is exactly represented by the exact effect (orthogonal projection) $E_{M_0(G)}$ which projects onto this common certainly-no domain.

Therefore, in Hilbertian BZ-posets $\Sigma(\mathscr{H})$ identifications (3.1) and (3.4) assume now the form

$$[F]_{\text{JP}} \leftrightarrow E_{M_1(F)} \leftrightarrow M_1(F) \tag{6.5}$$

$$[G]_{\text{CN}} \leftrightarrow E_{M_0(G)} \leftrightarrow M_0(G) \tag{6.6}$$

where any property (resp., noperty) is identified, not only with the unique exact effect which tests it, but also with the common certainly-yes (resp., no) domain.

*Proposition 6.2.* The Hilbertian BZ-poset $\Sigma(\mathscr{H})$ of all generalized orthogonal projections is a quantum BZ-poset. Indeed, we have the following.

(QH-1) The set of all its exact elements is the set $\mathscr{E}(\mathscr{H})$ of all orthogonal projections, which has the structure of the orthomodular orthocomplemented atomic complete lattice $\langle \mathscr{E}(\mathscr{H}), \mathbb{O}, \leq, ' \rangle$ with respect to the ordering (or-H). In particular, we recall that

$$E_1 \leq E_2 \quad \text{iff} \quad M_1(E_1) \subseteq M_1(E_2)$$

and that

$$E' := \mathbb{1} - E = E_{\mathrm{Ran}(E)^{\perp}}$$

(QH-2) For any family of generalized orthogonal projections $\{F_j : j \in J\}$ there exists the generalized orthogonal projection

$$\Pi F_j := \tfrac{1}{2}[E_{M_1(J)} + (E_{M_0(J)})']$$

where

$$M_1(J) := \bigcap_{j \in J} M_1(F_j) \qquad \text{and} \qquad M_0(J) := \bigcap_{j \in J} M_0(F_j)$$

such that

$$\nu(\Pi F_j) = \wedge_j \, \nu(F_j) \qquad \text{and} \qquad \mu(\Pi F_j) = \wedge_j \, \mu(F_j)$$

According to the general theory, the Łukasiewicz three-valued BZ-poset $L_3(\Sigma(\mathscr{H}))$ induced from $\Sigma(\mathscr{H})$ by modality is a lattice; this lattice consists of all ordered pairs

$$(\nu(F), \mu(F)') = (E_{M_1(F)}, E_{M_0(F)}) \tag{6.7}$$

equipped with the following.

(1) The partial ordering:

(or-LQ) $(\nu(F), \mu(F)') \subseteq (\nu(G), \mu(G)')$ iff $M_1(F) \subseteq M_1(G)$ and $M_0(G) \subseteq M_0(F)$.

(2) The regular degenerate orthocomplementation:

(doc-LQ) $-(E_{M_1(F)}, E_{M_0(F)}) = (E_{M_0(F)}, E_{M_1(F)})$.

(3) The weak orthocomplementation:

(woc-LQ) $\sim(E_{M_1(F)}, E_{M_0(F)}) = (E_{M_0(F)}, E_{M_0(F)^\perp})$.

The set $L_3(\Sigma(\mathscr{H}))_e$ of exact elements from $L_3(\Sigma(\mathscr{H}))$ is the collection of all ordered pairs, for arbitrary $E \in \mathscr{E}(\mathscr{H})$,

$$(E, \mathbb{1} - E) \leftrightarrow E \tag{6.8}$$

The elements of $L_3(\Sigma(\mathscr{H}))\backslash L_3(\Sigma(\mathscr{H}))_e$ are called *m-iper effects* and the elements of $L_3(\Sigma(\mathscr{H}))_e$ *m-iper exact effects*, these being identified [see (6.8)] with orthogonal projections of $\mathscr{H}$.

The *extensional mapping* is thus the mapping

$$ext\colon \quad \Sigma(\mathscr{H}) \mapsto L_3(\Sigma(\mathscr{H}))$$

associating to any effect $F \in \Sigma(\mathscr{H})$ the corresponding extension

$$ext(F) := (E_{M_1(F)}, E_{M_0(F)})$$

Two effects are congruent by modality iff they have the same extension, and in our Hilbertian case this assumes the form

$$F \cong G \qquad \text{iff} \quad M_1(F) = M_1(G) \text{ and } M_0(F) = M_0(G)$$

and the *quantum g-properties* are usually denoted by (Cattaneo and Nisticò, 1985; Cattaneo *et al.*, 1988)

$$\begin{aligned}[F]_{\mathrm{FR}} = \{F_j\colon F \cong F_j\} &= \{F_j\colon M_1(F) = M_1(F_j) \text{ and } M_0(F) = M_0(F_j)\} \\ &= [F]_{\mathrm{JP}} \cap [F]_{\mathrm{CN}}\end{aligned} \tag{6.9}$$

Note that for any effect $F$ we have that $[F]_{\mathrm{FR}} \subseteq [F]_{\mathrm{JP}}$ and so the following mapping is well defined:

$$\Phi\colon \quad [F]_{\mathrm{FR}} \mapsto [F]_{\mathrm{JP}}$$

Taking into account the notion of generalized properties ($g$-properties) and the results of Sections 5 and 6, the cumulative diagram (5.4) has now the form

$$
\begin{array}{ccccc}
\text{effect} & & \text{iper effect} & & g\text{-property} \\
\hline
F \in \mathscr{F}(\mathscr{H}) & \xrightarrow{\text{ext}} & (E_{M_1(F)}, E_{M_0(F)}) & \longleftrightarrow & [F]_{\text{FR}} \in pr_g(\mathscr{H}) \\
\Big\downarrow \nu & & \Big\downarrow \Box & & \Big\downarrow \Phi \\
E \in \mathscr{E}(\mathscr{H}) & \longleftrightarrow & (E, \mathbb{1} - E) & \longleftrightarrow & [E]_{\text{JP}} \in pr(\mathscr{H}) \\
\hline
\text{event} & & \text{iper event} & & \text{property}
\end{array}
\tag{6.10}
$$

where $E = E_{M_1(F)}$. Lastly, according to (5.1), one can make the following further identification:

$$[F]_{\text{FR}} \equiv ([F]_{\text{JP}}, [F]_{\text{CN}}) \tag{6.11}$$

## 7. PRECLUSIVITY PROPOSITIONAL HILBERTIAN QUANTUM LOGICS

If we denote by $X(\mathscr{H}) := \mathscr{H} \backslash \{\underline{0}\}$ the collection of all nonzero vectors representing *preparation procedures* of a single sample of the physical entity described by the Hilbert space $\mathscr{H}$, the usual orthogonality relation on vectors is a *preclusivity* (i.e., irreflexive and symmetrical) relation such that $\langle X(\mathscr{H}), \perp \rangle$ turns out to be an *orthoframe* (Dalla Chiara and Giuntini, 1989). We recall that this orthogonality relation can be equivalently stated as the following binary relation of *physical separability*:

$$\psi \perp \varphi \qquad \text{iff} \quad \exists E \in \mathscr{E}(\mathscr{H}), \quad P(\psi, E) = 1, \quad \text{and} \quad P(\varphi, E) = 0$$

The set of all Hilbertian *quantum propositions* is defined as follows:

$$L_f(X(\mathscr{H}), \perp) := \{(M_1, M_0) : M_1, M_0 \in \mathscr{M}(\mathscr{H}), M_1 \perp M_0\}$$

i.e., it is the family of all pairs of mutually orthogonal subspaces of $\mathscr{H}$. For any proposition $p = (M_1(p), M_0(p))$, the subspace $M_1(p)$ [resp., $M_0(p)$] is the *certainly-true* (resp., *false*) domain; any $\psi \in M_1(p)/\{\underline{0}\}$ [resp., $M_0(p) \backslash \{\underline{0}\}$] describes a preparation (semantical world) in which the proposition is *true* (resp., *false*). Preparations from $\mathscr{H} \backslash (M_1(p) \cup M_0(p))$ represent a semantical world of *indeterminacy* for $p$.

The one-to-one mapping

$$L_3(\Sigma(\mathscr{H})) \mapsto L_f(X(\mathscr{H}), \perp), (E_{M_1(F)}, E_{M_0(F)}) \to (M_1(F), M_0(F))$$

allows the identification

$$\begin{aligned} L_3(\Sigma(\mathcal{H})) &\equiv L_f(X(\mathcal{H}), \perp) \\ (E_{M_1(F)}, E_{M_0(F)}) &\leftrightarrow (M_1(F), M_0(F)) \end{aligned} \tag{7.1}$$

in such a way that the BZ lattice structure of $L_3(\Sigma(\mathcal{H}))$ is naturally transferred to $L_f(X(\mathcal{H}), \perp)$. Precisely, we have the BZ lattice of all Hilbertian quantum propositions characterized by the following.

(1) The partial ordering:

(or-PH) $(M_1(p), M_0(p)) \sqsubseteq (M_1(q), M_0(q))$ iff $M_1(p) \subseteq M_1(q)$ and $M_0(q) \subseteq M_0(p)$.

(2) The regular degenerate orthocomplementation:

(doc-PH) $\neg(\mathrm{M}_1, \mathrm{M}_0) = (\mathrm{M}_0, \mathrm{M}_1)$.

(3) The weak orthocomplementation:

(woc-PH) $\approx(M_1, M_0) = (M_0, M_0^{\perp})$.

The BZ lattice $L_e(X(\mathcal{H}), \perp)$ of all *quantum exact propositions* is

$$L_e(X(\mathcal{H}), \perp) := \{(M, M^{\perp}) : M \in \mathcal{M}(\mathcal{H})\}$$

which allows the identification

$$\begin{aligned} L_e(X(\mathcal{H}), \perp) &\equiv \mathcal{M}(\mathcal{H}) \\ (M, M^{\perp}) &\leftrightarrow M \end{aligned} \tag{7.2}$$

In this way any quantum exact proposition is identified with a subspace of the Hilbert space $\mathcal{H}$. The propositions which are not exact are called *quantum fuzzy propositions*. We recall the following results.

(NP) The necessity and possibility operators, written $\square$ and $\diamond$, respectively, are given by

$$\square(p) = \approx(\neg p) = (M_1(p), M_0(p)^{\perp}) \qquad \text{(necessity)}$$

$$\diamond(p) = \neg(\approx p) = (M_0(p)^{\perp}, M_0(p)) \qquad \text{(possibility)}$$

(QP1) The g.l.b. and the l.u.b., denoted by $\sqcap$ and $\sqcup$, respectively, of any family of quantum propositions $\{p_j = (M_1^{(j)}, M_0^{(j)}) : j \in J\}$ are given by

$$\sqcap(M_1^{(j)}, M_0^{(j)}) = (\cap M_1^{(j)}, \vee M_0^{(j)})$$

$$\sqcup(M_1^{(j)}, M_0^{(j)}) = (\vee M_1^{(j)}, \cap M_0^{(j)})$$

where $\vee M^{(j)} := (\cup M^{(j)})^{\perp\perp}$ denotes the subspace generated by the set-theoretic union of the subspaces $M^{(j)}$.

(QP2) For any family $\{p_j = (M_1^{(j)}, M_0^{(j)})\}$ of quantum propositions we define their "product" as the proposition

$$\Pi p_j := (\cap M_1^{(j)}, \cap M_0^{(j)})$$

Summarizing, we have the following graph:

$$\begin{array}{c} (M_1, M_0) \in L_f(X(\mathscr{H}), \perp) \\ \cup\!\mid \\ (M, M^{\perp}) \in L_e(X(\mathscr{H}), \perp) \equiv \mathscr{M}(\mathscr{H}) \ni M \end{array} \tag{7.3}$$

and so we can conclude as follows:

The orthomodular lattice $\mathscr{M}(\mathscr{H})$ of all Hilbertian subspaces is identified with the orthomodular "logic" $L_e(X(\mathscr{H}), \perp)$ of all Hilbertian exact quantum propositions, the latter being embedded into the "preclusivity logic" $L_f(X(\mathscr{H}), \perp)$ of all Hilbertian quantum propositions.

Owing to identification (7.1), the extensional mapping can be considered as a mapping

$$ext: \quad \Sigma(\mathscr{H}) \mapsto L_f(X(\mathscr{H}), \perp)$$

given by the law

$$F \in \Sigma(\mathscr{H}) \to ext(F) = (M_1(F), M_0(F))$$

the latter being a quantum proposition. Hence, there are several effects which have the same extension, i.e., which test the same quantum proposition.

The restriction of this extensional mapping to the set of all exact effects $\mathscr{E}(\mathscr{H})$

$$E \in \mathscr{E}(\mathscr{H}) \to ext(E) = (M_1(E), M_1(E)^{\perp})$$

is one-to-one and onto the quantum logic $L_e(X(\mathscr{H}), \perp)$.

In conclusion, we may make the following statement.

Any quantum proposition of the quantum BZ-logic $L_f(X(\mathscr{H}), \perp)$, in general, is experimentally tested by several different effects from $\Sigma(\mathscr{H})$, whereas any proposition of the corresponding exact quantum logic $L_e(X(\mathscr{H}), \perp) \equiv \mathscr{M}(\mathscr{H})$ is experimentally tested by a unique exact effect from $\mathscr{E}(\mathscr{H})$.

Moreover, diagram (6.8) can now be restated as follows

$$\begin{array}{ccccc} \text{effect} & & \text{proposition} & & g\text{-property} \\ \hline F\in\mathscr{F}(\mathscr{H}) & \xrightarrow{\text{ext}} & (M_1(F), M_0(F)) & \longleftrightarrow & [F]_{\mathrm{FR}}\in pr_g(\mathscr{H}) \\ \Big\downarrow{\scriptstyle \nu} & & \Big\downarrow{\scriptstyle \square} & & \Big\downarrow{\scriptstyle \Phi} \\ E\in\mathscr{E}(\mathscr{H}) & \longleftrightarrow & (M, M^{\perp})\equiv M & \longleftrightarrow & [E]_{\mathrm{JP}}\in pr(\mathscr{H}) \\ \hline \text{event} & & \text{exact prop.} & & \text{property} \end{array} \tag{7.4}$$

where $M$ is the subspace of $\mathscr{H}$ on which $E$ projects.

## 8. HILBERTIAN QUANTUM LOCALIZATION EFFECTS

In this section we consider the conventional quantum description of a particle in a one-dimensional space based on the Hilbert space $L_2(\mathbb{R})$ (Cattaneo and Nisticò, 1991). Any Borel measurable function $\omega: \mathbb{R} \mapsto [0, 1]$ represent a (macroscopic) classical localization effect, for which we introduce the Borel subsets

$$\Delta_1(\omega) := \omega^{-1}(\{1\}) \qquad \text{and} \qquad \Delta_0(\omega) := \omega^{-1}(\{0\})$$

[We also set $\Delta_p(\omega) := \omega^{-1}(0, 1] = \mathbb{R}\backslash\Delta_0(\omega)$ in the sequel.]

In particular, characteristic functions $\chi_\Delta$ of Borel subsets $\Delta$ of $\mathbb{R}$ are classical exact localization effects.

For any classical localization effect $\omega$, the linear operator

$$F(\omega): L_2(\mathbb{R}) \mapsto L_2(\mathbb{R})$$

is defined, for any $f \in L_2(\mathbb{R})$, as follows:

$$(F(\omega)f)(x) := \omega(x)f(x)$$

which is trivially a Hilbertian quantum effect, called a *quantum localization effect*, whose certainly-yes and certainly-no domains are, respectively,

$$M_1(F(\omega)) = \{\psi \in L_2(\mathbb{R}): \text{supp}(\psi) \subseteq \Delta_1(\omega)\} \tag{8.1}$$

$$M_0(F(\omega)) = \{\varphi \in L_2(\mathbb{R}): \text{supp}(\varphi) \subseteq \Delta_0(\omega)\} \tag{8.2}$$

To any classical exact localization effect $\chi_\Delta$ there corresponds the Hilbertian exact localization effect (orthogonal projection) $E(\Delta) := F(\chi_\Delta)$ defined as

$$(E(\Delta)f)(x) := \chi_\Delta(x)f(x)$$

For any quantum localization effect $F(\omega)$ we have the following.

(doc-Loc) The fuzzy orthocomplement $F(\omega)' \in \Sigma(L_2(\mathbb{R}))$ is the quantum effect

$$(F(\omega)'f)(x) := [1 - \omega(x)]f(x)$$

(woc-Loc) The intuitionistic orthocomplement $F(\omega)^{\sim} \in \Sigma(L_2(\mathbb{R}))$ is the quantum effect

$$(F(\omega)^{\sim} f)(x) := (\chi_{\Delta_0(\omega)})(x) f(x)$$

The necessity and possibility of the quantum localization effect $F(\omega)$ are the quantum exact localization effects, respectively,

$$(\nu(F(\omega))f)(x) = \chi_{\Delta_1(\omega)}(x) f(x), \qquad (\mu(F(\omega))f)(x) = \chi_{\Delta_p(\omega)}(x) f(x)$$

Two Hilbertian localization questions $F(\omega_1)$ and $F(\omega_2)$ are equivalent with respect to necessity iff $\Delta_1(\omega_1) = \Delta_1(\omega_2)$. Then any Hilbertian localization property is characterized by a Borel subset $\Delta_1$ and is the quantum property

$$l_1(\Delta_1) = \text{“the particle is (necessarily) localized in } \Delta_1\text{”}$$

Hence,

$$l_1(\Delta_1) \equiv \{F(\omega) : \Delta_1(\omega) = \Delta_1\} = [E(\Delta_1)]_{\mathrm{JP}} \tag{8.3}$$

The quantum property $l_1(\Delta_1)$ is sharply tested by the Hilbertian exact effect described by the orthogonal projection $E(\Delta_1)$; any Hilbertian localization effect $F(\omega)$, for which $\Delta_1(\omega) = \Delta_1$, measures in an unsharp way the same property.

The orthogonal projection $E(\Delta_1)$ projects onto the subspace of all $f \in L_2(\mathbb{R})$ whose support is in $\Delta_1$, and so nonzero vectors of this subspace represent preparation procedures in which this quantum property is actual (i.e., true with certainty); each single individual sample prepared according to any of these preparation procedures gives with certainty the answer “yes” to quantum property $l_1(\Delta_1)$, and the latter is an “element of reality” in this preparation.

The orthogonal projection $E(\Delta_1)'$ projects onto the subspace of all $g \in L_2(\mathbb{R})$ whose support is in $(\Delta_1)^c = \mathbb{R} \setminus \Delta_1$. Hence, if $l_1(\Delta_1) \equiv [E(\Delta_1)]_{\mathrm{JP}}$ is the quantum property “the particle is localized in $\Delta_1$,” then

$$\neg l_1(\Delta_1) \equiv [E(\Delta_1)']_{\mathrm{JP}} = [E((\Delta_1)^c)]_{\mathrm{JP}} \equiv l_1((\Delta_1)^c) \tag{8.4}$$

That is, the quantum property “the particle is not localized in $\Delta_1$” is just the quantum property “the particle is localized in $(\Delta_1)^c$.” Each nonzero vector from $L_2(\mathbb{R})$ whose support is contained in $(\Delta_1)^c$ represents a preparation procedure of single individual samples for which the quantum property $\neg l_1(\Delta)$ is true with certainty (i.e., it is an “element of reality”).

*Remark 1.* Note that if $F(\omega)$ is a quantum localization effect which unsharply tests the property of being localized in $\Delta_1$ [i.e., $\Delta_1(\omega)=\Delta_1$], then $F(\omega)'$ is not an effect which unsharply tests the property of being localized in $(\Delta_1)^c$ [i.e., $\Delta_0(\omega)\neq(\Delta_1)^c$]. The quantum exact localization effect $E(\Delta_1)$ is the unique quantum effect with this property.

Two Hilbertian localization questions $G(\omega_1)$ and $G(\omega_2)$ are equivalent with respect to impossibility iff $\Delta_0(\omega_1)=\Delta_0(\omega_2)$. Then any Hilbertian localization noperty is characterized by a Borel subset $\Delta_0$ and is the quantum noperty

$$l_0(\Delta_0)=\text{"it is impossible to localize the particle in } \Delta_0\text{"}$$

Hence,

$$l_0(\Delta_0)\equiv\{G(\omega):\Delta_0(\omega)=\Delta_0\}=[E(\Delta_0)]_{CN} \tag{8.5}$$

The quantum noperty $l_0(\Delta_0)$ is sharply tested by the Hilbertian exact effect described by the orthogonal projection $E(\Delta_0)$; any Hilbertian localization effect $G(\omega)$, for which $\Delta_0(\omega)=\Delta_0$, measures in an unsharp way the same noperty.

Two Hilbertian localization questions $F(\omega_1)$ and $F(\omega_2)$ are congruent by modality iff $\Delta_1(\omega_1)=\Delta_1(\omega_2)$ and $\Delta_0(\omega_1)=\Delta_0(\omega_2)$. Then any Hilbertian localization generalized property is characterized by a pair of Borel subsets $(\Delta_1, \Delta_0)$ (with $\Delta_1\cap\Delta_0=\varnothing$) and is the quantum $g$-property

$l_g(\Delta_1, \Delta_0)=$ "the particle is necessarily localized in $\Delta_1$ and it is impossible to localize the particle in $\Delta_0$"

and, according to (6.11), this localization $g$-property is identified with the pair

$$l_g(\Delta_1, \Delta_0)\equiv(l_1(\Delta_1), l_0(\Delta_0)) \tag{8.6}$$

Hence, we can affirm the following in the case $\Delta_0\neq(\Delta_1)^c$:

> All quantum localized effects $F(\omega_j)\in[F(\omega)]_{FR}$ describe effects which, relative to a localization observation, unsharply test if the particle is localized in an (unknown) Borel region $\Delta$ of the real line such that
>
> $$\Delta_1\subseteq\Delta\subseteq\Delta_p=\mathbb{R}\setminus\Delta_0$$

According to Proposition 5.1, the quantum localization exact property $l_e(\Delta_1, \Delta_1^c)\equiv l_1(\Delta_1)$ is sharply tested by the unique quantum localization exact effect $E(\Delta_1)$.

The extensional mapping applies to quantum localization effects, giving a corresponding quantum localization proposition in the following way:

$$F(\omega) \mapsto ext(F(\omega))$$
$$= (\{\psi \in L_2(\mathbb{R}) : \mathrm{supp}(\psi) \subseteq \Delta_1(\omega)\}, \{\varphi \in L_2(\mathbb{R}) : \mathrm{supp}(\varphi) \subseteq \Delta_0(\omega)\})$$

Nonzero vectors from

$$\{\psi \in L_2(\mathbb{R}) : \mathrm{supp}(\psi) \subseteq \Delta_1(\omega)\}$$

[resp., $\{\varphi \in L_2(\mathbb{R}) : \mathrm{supp}(\varphi) \subseteq \Delta_0(\omega)\}$] describe preparation procedures in which the elementary statement "the particle is localized in $\Delta$, when the latter is tested by the classical localization device $\omega$" is "true" (resp., "false").

## REFERENCES

Cattaneo, G., and Marino, G. (1988). Non-usual orthocomplementations in partially ordered sets and fuzziness, *Fuzzy Sets and Systems*, **25**, 107.

Cattaneo, G., and Nisticò, G. (1985). Complete effect-preparation structures, *Nuovo Cimento*, **90B**, 161.

Cattaneo, G., and Nisticò, G. (1989). Brouwer–Zadeh posets and three valued Łukasiewicz posets, *Fuzzy Sets and Systems*, **33**, 165.

Cattaneo, G., and Nisticò, G. (1991). Axiomatic foundations of quantum physics: critiques and misunderstandings. Piron's question-proposition system, *International Journal of Theoretical Physics*, **30**, 1293.

Cattaneo, G., Dalla Pozza, C., Garola, C., and Nisticò, G. (1988). On the logical foundations of the Jauch Piron approach to quantum physics, *International Journal of Theoretical Physics*, **27**, 1313.

Cattaneo, G., Garola, C., and Nisticò, G. (1989). Preparation-effect versus question-propositions structures, *Physics Essays*, **2**, 197.

Dalla Chiara, M., and Giuntini, R. (1989). Paraconsistent quantum logics, *Foundations of Physics*, **19**, 891.

Giuntini, R. (1990). Brouwer–Zadeh logic and the operational approach to quantum mechanics, *Foundations of Physics*, **20**, 701.

Jauch, J. M., and Piron, C. (1969). On the structure of quantal proposition systems, *Helvetica Physica Acta*, **42**, 842.

Kraus, K. (1983). *States, Effects, and Operations* (Lecture Notes in Physics, Vol. 190), Springer-Verlag, Berlin.

Ludwig, G. (1983). *Foundations of Quantum Mechanics*, Vol. I, Springer-Verlag, Berlin.

Piron, C. (1976). *Foundations of Quantum Physics*, Benjamin, Reading, Massachusetts.